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THE (φ, 1) RECTIFIABLE SUBSETS OF EUCLIDEAN SPACE

BY

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ABSTRACT. In this paper the structure of a subset $E \subset \mathbf{R}^n$ with $\mathbf{H}^1(E) < \infty$ has been studied by examining its intersection with various translated positions of a smooth hypersurface B. The following result has been established:

Let B be a proper $(n-1)$ dimensional smooth submanifold of $\mathbf{R}^n$ with nonzero Gaussian curvature at every point. If $E \subset \mathbf{R}^n$ with $\mathbf{H}^1(E) < \infty$, then there exists a countably 1-rectifiable Borel subset R of $\mathbf{R}^n$ such that $(E \sim R)$ is purely $(\mathbf{H}^1, 1)$ unrectifiable and $(E \sim R) \cap (g + B) = \emptyset$ for almost all $g \in \mathbf{R}^n$.

Furthermore, if in addition E is $\mathbf{H}^1$ measurable and $E \cap (g + B) = \emptyset$ for $\mathbf{H}^n$ almost all $g \in \mathbf{R}^n$ then $\mathbf{H}^1(E \cap R) = 0$. Consequently, E is purely $(\mathbf{H}^1, 1)$ unrectifiable.

Introduction. The study of the geometric structure of subsets of $\mathbf{R}^n$ relative to properties of their projections on k-dimensional linear subspaces has always played a central role in the progress of geometric measure theory. For example, the proof in [FF] of the existence of solutions for Plateau's problem and the minimal surface problem is dependent on this structure theory. The first results in this direction were obtained by Besicovitch in [BE], where he characterized 1-dimensional rectifiable subsets of $\mathbf{R}^2$ in terms of their projection properties. His results were extended by Federer [F2] to subsets of $\mathbf{R}^n$ and by Brothers [B] to subsets of homogeneous spaces.

Federer showed that if $E \subset \mathbf{R}^n$ with $\mathbf{H}^k(E) < \infty$ then there exists a countably k-rectifiable Borel subset R of $\mathbf{R}^n$ such that $E \sim R$ is purely $(\mathbf{H}^k, k)$ unrectifiable and $\mathbf{L}^k[p(E \sim R)] = 0$ for almost all orthogonal projections $p: \mathbf{R}^n \to \mathbf{R}^k$ where $\mathbf{L}^k$ is the Lebesgue measure in $\mathbf{R}^k$.

Brothers generalized Federer's results to subsets of a smooth n-dimensional Riemannian manifold X with a transitive group of isometries G. In order to make the transition from $\mathbf{R}^n$ to X it was necessary to restate the projection properties without referring to projections. This he achieved by replacing

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[1]This research contains the author's main result in his Ph.D. dissertation at Indiana University (1975).

orthogonal projections of $A \subset \mathbf{R}^n$ into $\mathbf{R}^k$ with intersections $A \cap g(P)$ where g is an isometry of $\mathbf{R}^n$ and P a fixed $(n-k)$-dimensional linear subspace. For example the statement "$p(A)$ has Lebesgue measure zero for almost all orthogonal projections $p\colon \mathbf{R}^n \to \mathbf{R}^k$" is equivalent to "$A \cap g(P)$ is empty for almost all isometries g." Thus his main result has the following form:

Let G be a Lie group of isometries of X with $\dim G = n(n+1)/2$ and suppose G acts transitively on X. Let B be a fixed $(n-k)$-dimensional smooth submanifold of X. If $E \subset X$ with $\mathbf{H}^k(E) < \infty$ then there exists a countably k-rectifiable Borel subset R of X such that $(E \sim R)$ is purely $(\mathbf{H}^k, k)$ unrectifiable and

$$(E \sim R) \cap g(B) = \varnothing$$

for almost all $g \in G$.

One of the central features of the proof of the above theorem is the use of the fact that the isotropy group at a point $0 \in X$ acts on the tangent space at 0 as either the orthogonal group or the special orthogonal group. We also note that $\dim G = n(n+1)/2$ implies in the Euclidean case that G is either the full group of isometries or the component of this group which contains the identity. Further, if X is connected and $\dim G = n(n+1)/2$, then X must be of constant curvature. Thus it is natural to ask if it is possible to obtain similar results with less restrictive assumptions on G; that is, can Brothers' results hold if the dimension of G is less than $n(n+1)/2$?

Notice that if we take $k = 1$ and $B = \mathbf{S}^{n-1} \subset \mathbf{R}^n$ in Brothers' theorem then it follows that for almost all translations g of $\mathbf{R}^n$

$$(E \sim R) \cap g(B) = \varnothing.$$

On the other hand, standard examples (see for example [**F1**, 3.3.19]) show that this may not be true if B is a hyperplane. Based upon these examples together with the structure of the proof of his theorem, Brothers conjectured that if G is the group of translations of $X = \mathbf{R}^n$ then his result will hold at least for $(\mathbf{H}^1, 1)$ rectifiability provided it is assumed that the Gaussian curvature of B does not vanish. In this paper we prove this conjecture. Our main result is the following:

THEOREM 1. *Let B be a proper $(n-1)$-dimensional smooth submanifold of $\mathbf{R}^n$ with nonzero Gaussian curvature at every point. If $E \subset \mathbf{R}^n$ with $\mathbf{H}^1(E) < \infty$, then there exists a countably 1-rectifiable Borel subset R of $\mathbf{R}^n$ such that $(E \sim R)$ is purely $(\mathbf{H}^1, 1)$ unrectifiable and*

$$(E \sim R) \cap (g + B) = \varnothing$$

for almost all $g \in \mathbf{R}^n$.

Furthermore, if in addition E is $\mathbf{H}^1$ measurable and $E \cap (g + B) = \varnothing$ for

$\mathbf{H}^n$ *almost all* $g \in \mathbf{R}^n$ *then* $\mathbf{H}^1(E \cap R) = 0$. *Consequently, E is purely* $(\mathbf{H}^1, 1)$ *unrectifiable.*

Theorem 2 is an extension of this result involving measures more general than $\mathbf{H}^1$.

The problem of extending these results to the general case where X is a Lie group with an invariant metric is difficult because of noncommutativity. On the other hand, our results clearly hold when X is a torus, hence for the case where X is an Abelian Lie group. In a subsequent paper we will investigate the possibility of extending our results to the case where $k > 1$.

I am indebted to Professor John Brothers for his continuing help, advice and encouragement during the preparation of this paper. He has always been a source of inspiration.

Preliminaries. The purpose of this section is to fix basic notation and terminology; more details may be found in [F1].

If M is an l-dimensional manifold of class 1 and $u \in M$, then $\mathbf{T}_u(M)$ is the l-dimensional real vector space of *tangent vectors of M at u.*

For each finite dimensional vector space V and $l = 0, 1, 2, \ldots, \dim V$, $\Lambda_l(V)$ is the associated vector space of l vectors. Furthermore,

$$\Lambda_*(V) = \bigoplus_{l=0}^{\dim V} \Lambda_l(V)$$

is the corresponding exterior algebra, with exterior multiplication $\wedge$.

Suppose M and N are manifolds of class 1 and $f\colon M \to N$. If $u \in M$, $w = f(u)$ and f is differentiable at u, *the differential of f at u* is a linear transformation

$$f_\#(u)\colon \mathbf{T}_u(M) \to \mathbf{T}_w(N).$$

$f_\#(u)$ can be extended to a unique algebra homomorphism

$$f_\#(u)\colon \Lambda_*[\mathbf{T}_u(M)] \to \Lambda_*[\mathbf{T}_w(N)].$$

If M and N are Riemannian manifolds and

$$r = \inf\{\dim M, \dim N\}$$

then the *Jacobian of f at u* is

$$Jf(u) = \sup\{|f_\#(u)(v)|\colon v \in \Lambda_r[\mathbf{T}_u(M)], |v| = 1\}$$

where the indicated norm is induced by the metric on M and N.

If $u = (u_1, \ldots, u_n)$ and $w = (w_1, \ldots, w_n) \in \mathbf{R}^n$, the inner product of u and w is $u \cdot w = \sum_{i=1}^n u_i \cdot w_i$.

$e_1, \ldots, e_n$ are the *standard orthonormal basis vectors of* $\mathbf{R}^n$.

Let ϕ be a nonnegative measure on a Riemannian manifold M such that closed sets are ϕ measurable. In particular, $\mathbf{H}^l$ is the *l-dimensional Hausdorff measure on M.*

The family of Suslin subsets of M contains the Borel subsets of M and has the following properties:

Each Suslin set is ϕ measurable.

If $\{F_i\}$ is a countable family of Suslin sets, then $\bigcup_{i=1}^{\infty} F_i$ and $\bigcap_{i=1}^{\infty} F_i$ are Suslin sets.

If N is a smooth manifold and f: $M \to N$ is continuous, then $f(S)$ and $f^{-1}(T)$ are Suslin sets whenever S and T are Suslin subsets of M and N respectively.

If μ measures Y and $A \subset Y$, then $\mu \llcorner A$ is the measure on Y defined by the formula

$$\mu \llcorner A(S) = \mu(A \cap S) \quad \text{for } S \subset Y.$$

If f: $Y \to Z$, then $f_{\#}(\mu)$ is the measure on Z defined by

$$f_{\#}(\mu)(S) = \mu[f^{-1}(S)] \quad \text{for } S \subset Z.$$

$R \subset M$ is *k-rectifiable* if there exists a Lipschitzian function mapping some bounded subset of $\mathbf{R}^k$ onto R.

$R \subset M$ is *countably k-rectifiable* if R is the union of a countable family of k-rectifiable sets.

$E \subset M$ is *countably* (ϕ, k) *rectifiable* if there exists a countably k-rectifiable set R with $\phi(E \sim R) = 0$.

$E \subset M$ is (ϕ, k) *rectifiable* if $\phi(E) < \infty$ and E is countably (ϕ, k) rectifiable.

$E \subset M$ is *purely* (ϕ, k) *unrectifiable* if E contains no k-rectifiable set R with $\phi(R) > 0$.

$$\mathbf{U}_k(u, r) = \mathbf{R}^k \cap \{w: |w - u| < r\}$$

for $r > 0$, $u \in \mathbf{R}^k$.

If $r > 0$, $s > 0$, $u \in \mathbf{R}^n$ and $Y \subset \mathbf{R}^n$, then

$$\mathbf{X}(u, r, Y, s) = \mathbf{R}^n \cap \{w: \operatorname{dist}(w, Y) < s \operatorname{dist}(w, u)\} \cap \mathbf{U}_n(u, r).$$

Throughout this paper B will denote a proper $(n-1)$-dimensional submanifold of class ∞ of $\mathbf{R}^n$ with nonzero Gaussian curvature at every point. If $s > 0$ and $g \in -B$ denote

$$K_{g,s} = \{h: |h - g| < s\} \cap (-B).$$

Also,

$$K_{g,s}(B) = \{h + b: h \in K_{g,s}, b \in B\}.$$

If F: $U \to \mathbf{R}$ is a C^2 function, where U is an open subset of $\mathbf{R}^n$, we denote

$$D_i F(x) = \partial F(x)/\partial x_i, \qquad i = 1, 2, \ldots, n.$$

$$D_{ij}F(x) = \partial^2 F(x)/\partial x_i \partial x_j, \qquad i, j = 1, 2, \dots, n.$$

We will denote by f_0: $\mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^n$ the map defined by

$$f_0(a, b) = a - b.$$

If $A \subset \mathbf{R}^n$, then $S_{A,1}$ is the set of $(a, b) \in \mathbf{R}^n \times B$ such that for some $\varepsilon > 0$

$$\lim_{s \to 0^+} \sup_{0<r<\varepsilon} \phi\left[A \cap \mathbf{X}(a, r, a - b + B, s)\right](rs)^{-1} = 0.$$

$S_{A,2}$ is the set of $(a, b) \in \mathbf{R}^n \times B$ such that for all $\varepsilon > 0$

$$\lim_{s \to 0^+} \sup_{0<r<\varepsilon} \phi\left[A \cap \mathbf{X}(a, r, a - b + B, s)\right](rs)^{-1} = \infty.$$

$S_{A,3} = \mathbf{R}^n \times B \cap \{(a, b): a \in \operatorname{Clos}[A \cap (a - b + B) \sim \{a\}]\}$.

LEMMA 1. *For any $g \in -B$ there exist positive constants r_1, s_1, α, β such that if $0 < s < s_1$, then*

(i) $\mathbf{X}(0, r_1, g + B, s) \subset K_{g,\alpha s}(B)$,

(ii) $\mathbf{X}(0, \infty, g + B, s) \supset K_{g,\beta s}(B) \sim \{0\}$.

PROOF. Without loss of generality we may assume $g = 0 \in B$. By a proper choice of coordinate axes we may assume that

$$B \cap \mathbf{U}_n(0, 1) = \{(x, f(x)): x \in U\},$$

where $0 \in U$, U is an open subset of $\mathbf{R}^{n-1}$ which contains $\{x \in \mathbf{R}^{n-1}: |x| < 1/2\}$ and f: $U \to \mathbf{R}$ is of class C^∞ and such that

$$(1) \qquad f(0) = 0, \qquad D_i f(0) = 0 \quad \text{for } i = 1, 2, \dots, n - 1.$$

We may also assume that f is Lipschitzian and if $|x| \leqslant 1/2$, then

$$(2) \qquad |f(x)| \leqslant |x|,$$

$$(3) \qquad |D_j f(x)|, |D_{jl} f(x)|, |D_{jlm} f(x)| < Cn^{-3}$$

for $j, l, m = 1, 2, \dots, n - 1$, where $C > 1$.

Our assumption that the Gaussian curvature does not vanish at any point of B leads to the fact [**KN**, Volume 2, p. 17] that

$$(4) \qquad \det(D_{lj} f(0)) \neq 0, \qquad l, j = 1, 2, \dots, n - 1.$$

Let L be the linear transformation of $\mathbf{R}^{n-1}$ with the matrix $(D_{lj} f(0))$. Then there exists $0 < V \leqslant C/2$ such that whenever $y \in \mathbf{S}^{n-2} \subset \mathbf{R}^{n-1}$,

$$(5) \qquad |L(y) \cdot e_j| \geqslant V \quad \text{for some } j \in \{1, 2, \dots, n - 1\}.$$

PART 1. *Fix $x, x_0 \in \mathbf{R}^{n-1}$ and $s \in \mathbf{R}$ such that $0 < |x_0| < (V/8C)$, $|x - x_0| < (sV|x_0|/16C)$, and $0 < s < (V/8Cn)$. Then the set of numbers $f(x + t) - f(t) - f(x_0)$ corresponding to $t = t_l e_l$ and $l \in \{1, 2, \dots, n - 1\}$ with $|t_l| \leqslant s$ contains the interval $\{r: |r| < sV|x_0|/16\}$.*

PROOF. Let us consider the sets

$$S_1 = \{x: 0 < |x| < (V/4C)\} \cap \mathbf{R}^{n-1},$$
$$S_2 = \{t: 0 < |t| < (V/2C)\} \cap \mathbf{R}^{n-1}.$$

Let $x \in S_1$, $t \in S_2$. Expanding $f(x + t)$ about x and $f(t)$ about 0, we have

$$f(x + t) - f(t) = f(x) + Df(x)(t) + \sum_{j,l=1}^{n-1} t_j t_l R_{jl}, \tag{6}$$

where

$$R_{jl} = \int_0^1 (1 - \theta)\{D_{jl}f(x + \theta t) - D_{jl}f(\theta t)\}\, d\theta.$$

By assumption (3)

$$|R_{jl}| \leqslant |x| C/2. \tag{7}$$

Also,

$$Df(x)(t) = \sum_{j=1}^{n-1} t_j \left[\sum_{l=1}^{n-1} x_l D_{jl} f(0) + \frac{1}{2} \sum_{m,l=1}^{n-1} x_m x_l D_{jml} f(\theta_j x) \right] \tag{8}$$

where $0 \leqslant \theta_j \leqslant 1$.

Now let $y = (x/|x|) \in \mathbf{S}^{n-2}$. Then from (5), for some l

$$0 < V \leqslant \left| \sum_k (x_k/|x|) D_{lk} f(0) \right|. \tag{9}$$

Consequently by setting $t_j = 0$ for $j \neq l$ in (8), we get

$$t_l D_l f(x) = |x| t_l H, \tag{10}$$

where

$$H = \sum_k (x_k/|x|) D_{lk} f(0) + (|x|/2) \sum_{k,m} (x_m/|x|)(x_k/|x|) D_{lmk} f(\theta_l x).$$

By (9) and (3), we conclude that

$$|H| \geqslant V - V/8 > V/2.$$

Therefore from (6), (10) we get

$$\begin{aligned} &f(x + t) - f(t) - f(x) = |x| t_l H + t_l^2 R_{ll} \text{ with} \\ &|H| > V/2 \text{ and } |R_{ll}| < (|x| C/2) \text{ for} \\ &t = t_l e_l \in S_2 \text{ and } x \in S_1. \end{aligned} \tag{11}$$

Now let x, x_0, s be such that $0 < |x_0| \leqslant (V/8C)$, $|x - x_0| < (sV|x_0|/16C)$ and $0 < s < (V/8Cn)$. Notice that $x \in S_1$. Also,

$$|f(x) - f(x_0)| \leqslant (sV|x_0|/16). \tag{12}$$

Now by (11), for $t = t_l e_l \in S_2$,

$$f(x+t) - f(t) - f(x_0) = |x| t_l H + t_l^2 R_{ll} + f(x) - f(x_0),$$

where $|H| > V/2$ and $|R_{ll}| \leq |x| C/2$.

Suppose $H > 0$. Since $s < V/2C$, $t = se_l \in S_2$. By setting $t_l = s$, we obtain

$$\begin{aligned} f(x+t) - f(t) - f(x_0) &\geq (|x|/2)(Vs - Cs^2) - sV|x_0|/16 \\ &\geq (|x_0|/2)(1/2)s(V - Cs) - sV|x_0|/16 \geq sV|x_0|/16. \end{aligned}$$

Similarly, putting $t_l = -s$, we obtain

$$f(x+t) - f(t) - f(x_0) \leq -sV|x_0|/16.$$

Since for fixed $x, x_0, s, f(x+t) - f(t) - f(x_0)$ with $t = t_l e_l$ is a continuous function of t_l on the interval $[-s, s]$, we conclude that Part 1 holds. Finally, in the case where $H < 0$ we reach the same conclusion by replacing s by $-s$.

PART 2. *$K_{0,4s}(B \cap \mathbf{U}_n(0, 1)) \supset \mathbf{U}_n((x_0, f(x_0)), rVs/64C)$ where $0 < s < V/8Cn$, $0 < |x_0| < V/8C$ and $r = |(x_0, f(x_0))|$.*

PROOF. Let

$$\begin{aligned} S_{x_0} = \mathbf{R}^{n-1} &\cap \{x: |x - x_0| < sV|x_0|/16C\} \\ &\times \mathbf{R} \cap \{\xi: |\xi - f(x_0)| < (sV|x_0|/16)(1 - sV/8C)\}. \end{aligned}$$

Since $|x_0| < 1/2$, $r \leq 2|x_0|$ by (2). Also,

$$(Vs|x_0|/16)(1 - sV/8C) \geq rVs/64C.$$

Thus

$$\mathbf{U}_n\big((x_0, f(x_0)), rVs/64C\big) \subset S_{x_0}. \tag{13}$$

So, let $(x, \xi) \in S_{x_0}$. We observe that $x \in S_1$. Since $|\xi - f(x_0)| < sV|x_0|/16$, by Part 1 there exists $t = t_l e_l$ such that $|t| \leq s$, and

$$f(x+t) - f(t) - f(x_0) = \xi - f(x_0).$$

Notice that by (2) $(-t, -f(t)) \in K_{0,4s} \subset -B$. Also,

$$(x+t, f(x+t)) \in B \cap \mathbf{U}_n(0, 1).$$

Thus

$$(x, \xi) \in K_{0,4s}(B \cap \mathbf{U}_n(0, 1)).$$

This together with (13) establishes our claim.

PART 3. *$K_{0,4s}(B) \supset \mathbf{X}(0, V/16C, B, sV/128C)$ where $0 < x < V/8nC$.*

PROOF. If $w \in \mathbf{X}(0, V/16C, B, sV/128C)$, then there exists $b \in B$ such that $|b - w| < r_0 sV/128C$, where $r_0 = |w|$. Now $r = |b| \geq |w| - |b - w| \geq r_0/2$ and $r \leq 2r_0 \leq V/8C$. Therefore $b = (x_0, f(x_0))$ with $|x_0| < V/8C$. Hence

$$w \in \mathbf{U}_n\big((x_0, f(x_0)), rsV/64C\big) \subset K_{0,4s}(B \cap \mathbf{U}_n(0, 1))$$

by Part 2.

From Part 3 we get (i) of Lemma 1 immediately.

Turning to the proof of (ii) of the lemma, we claim that

$$K_{0,2s}(B) \sim \mathbf{U}_n(0, 1/16) \subset \mathbf{X}(0, \infty, B, 32s), \tag{14}$$

where $0 < s < V/8nC < 1/32$. Indeed, if $u \in K_{0,2s}(B) \sim \mathbf{U}_n(0, 1/16)$, then $u = g + w$; $g \in K_{0,2s}$, $w \in B$. But then

$$\big[\text{distance}(u, B)\big]/|u| \leqslant 32s.$$

Next let $u_1 \in K_{0,2s}(B) \cap \mathbf{U}_n(0, 1/16)$. This means that $u_1 = g_1 + w_1$, with $g_1 \in K_{0,2s}$ and $w_1 \in B$, and $|u_1| < 1/16$. Therefore

$$K_{0,2s}(B) \cap \mathbf{U}_n(0, 1/16) \subset K_{0,2s}(B \cap \mathbf{U}_n(0, 1/8)). \tag{15}$$

So let us assume $u \in K_{0,2s}(B \cap \mathbf{U}_n(0, 1/8))$. Then $u = (t, -f(-t)) + (x, f(x))$ where $(t, -f(-t)) \in K_{0,2s}$ and $(x, f(x)) \in B \cap \mathbf{U}_n(0, 1/8)$. Write $y = x + t$, so that $|y| < 1/4$. Thus $y \in U$. Also,

$$\begin{aligned} f(x) - f(-t) &= f(x) - f(x - y) \\ &= f(y) + Df(y)(x - y) + \sum_{j,l} (x_j - y_j)(x_l - y_l) R_{jl} \end{aligned}$$

where

$$R_{jl} = \int_0^1 (1 - \theta)\big\{D_{jl}f(y + \theta(x - y)) - D_{jl}f(\theta(x - y))\big\}\, d\theta$$

and, as in (7), $|R_{jl}| \leqslant |y|C/2$. Note that $|D_i f(y)| \leqslant C|y|$. We thus conclude that

$$|f(x) - f(-t) - f(y)| \leqslant C|y|2sn + n^2(2s)^2|y|C/2.$$

Writing $v = (y, f(y))$ we find that

$$|f(x) - f(-t) - f(y)| < 4|v|Cn^2 s, \tag{16}$$

hence distance$(u, B) \leqslant |u - v| < 4|v|Cn^2 s$. Since $|v| \leqslant 2|u|$ by (2) we conclude that $u \in \mathbf{X}(0, \infty, B, 8Cn^2 s)$. Hence

$$K_{0,2s}(B) \cap \mathbf{U}_n(0, 1/16) \subset \mathbf{X}(0, \infty, B, 8Cn^2 s),$$

by (15). This together with (14) establishes (ii).

LEMMA 2. *Let $g \in -B$. Then there exist positive numbers r_2, s_2, H_1, H_2 and $\delta > 1$ such that if $0 < s < s_2$ and $0 \neq w \in K_{g,s}(B) \cap \mathbf{U}_n(0, r_2)$, then*

(i) $\mathbf{H}^{n-2}[(w - B) \cap K_{g,\delta s}] \geqslant H_1 s^{n-2}$,

(ii) $\mathbf{H}^{n-2}[(w - B) \cap K_{g,s}] \leqslant H_2 s^{n-2}$.

PROOF. Assuming $g = 0 \in B$ and choosing f and U as in the proof of

Lemma 1 we denote $U_0 = -U$ and define $h: U_0 \to \mathbf{R}$ by $h(x) = -f(-x)$. Thus

$$-B \cap \mathbf{U}_n(0, 1) = \{(x, h(x)): x \in U_0\}.$$

Since $\det(D_{ij}h(0)) \neq 0$, we can (with the use of an orthogonal change of coordinates) assume h has a Taylor expansion of the form

$$h(x) = \sum_{i=1}^{n-1} k_i x_i^2 + \sum_{i,j,k=1}^{n-1} \alpha_{ijk}(x) x_i x_j x_k, \tag{1}$$

with $k_i \neq 0$, $|\alpha_{ijk}|$ and $|D_l \alpha_{ijk}|$ bounded and C^∞ on U_0. We will write

$$K_0 = \min(|k_1|, \ldots, |k_{n-1}|) > 0 \quad \text{and} \quad K = \max(1, |k_1|, \ldots, |k_{n-1}|).$$

For $i = 1, 2, \ldots, (n-1)$ we will write $\pi_i x = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{n-1}) \in \mathbf{R}^{n-2}$ whenever $x = (x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_{n-1}) \in \mathbf{R}^{n-1}$.

PART 1. *There exist $C > 2$, $0 < \gamma < \min(1/16, K_0/[4C(2n)^{1/2}])$, and a C^∞ function Φ defined on*

$$\Omega = \mathbf{R}^{n-1} \times \mathbf{R} \times \mathbf{R}^{n-1} \times \mathbf{R} \cap \{(x, \rho, y, \eta): |x| < 1/4, |\rho| < 1/4, |y|^2 + |\eta|^2 = 1\}$$

such that if $0 < s < \gamma$ and $0 \neq w = \rho(y, \eta) \in K_{0,s}(B) \cap \mathbf{U}_n(0, \gamma)$ with $|y|^2 + |\eta|^2 = 1$, then the following are true:

(i) $(w - B) \cap K_{0,s} = \{(x, h(x)) \in K_{0,s}: \Phi(x, \rho, y, \eta) = 0\}$.

(ii) *There exists $i \in \{1, 2, \ldots, n-1\}$ (which depends only on y) such that*

$$|y_i| > (2n)^{-1/2},$$

and for $|x| \leq \gamma$,

$$|D_i \Phi(x, \rho, y, \eta)| \geq 1/C$$

and

$$|D_j \Phi(x, \rho, y, \eta)/D_i \Phi(x, \rho, y, \eta)| \leq C \quad \text{for } j = 1, 2, \ldots, n-1, j \neq i.$$

(iii) *If for $i = 1, 2, \ldots, n-1$ we write $\Phi_i(x_i, \rho, y, \eta) = \Phi(x, \rho, y, \eta)$ where $x = (0, \ldots, 0, x_i, 0, \ldots, 0)$, $|x_i| < 1/4$, then there exist C^∞ functions ϕ_i, g_i, each having domain $\mathbf{R} \times \mathbf{R} \times \mathbf{R}^{n-1} \times \mathbf{R} \cap \{(x_i, \rho, y, \eta): |x_i|, |\rho| < 1/4, |y|^2 + |\eta|^2 = 1\}$ such that $|g_i|, |\phi_i|, |D_1 \phi_i| \leq C$ and*

$$\Phi_i(x_i, \rho, y, \eta) = 2k_i x_i y_i - \rho \sum_{i=1}^{n-1} k_i y_i^2 - \eta + g_i(x_i, \rho, y, \eta) x_i^2 + \rho \phi_i(x_i, \rho, y, \eta).$$

PROOF. By choosing ρ_0 sufficiently small ($0 < \rho_0 < 1/8$) we may assume that

$$B \cap \mathbf{U}_n(0, 2\rho_0) \sim \{0\} \subset \mathbf{X}(0, \infty, \mathbf{T}_0(B), 1/8).$$

Now if $u \in \mathbf{X}(0, \infty, B \cap \mathbf{U}_n(0, 2\rho_0), s_0)$ where $0 < s_0 < 1/8$ then there exists $0 \neq b \in B \cap \mathbf{U}_n(0, 2\rho_0)$ such that $|u - b| < |u|s_0$. But by the above assumption there exists $w \in \mathbf{T}_0(B)$ such that $|b - w| < |b|/8 < 9|u|/64$. Hence $u \in \mathbf{X}(0, \infty, \mathbf{T}_0(B), 1/2)$. Assuming $s_0 < s_1$, we apply Lemma 1(ii) with $\beta s = \rho_1 = \min(\beta s_0, \rho_0)$ to obtain

(2) $$\left[K_{0,\rho_1}(B \cap \mathbf{U}_n(0, 2\rho_1))\right] \sim \{0\} \subset \mathbf{X}(0, \infty, \mathbf{T}_0(B), 1/2).$$

Consider now $x \in \mathbf{R}^{n-1}$ and $(t, \theta) \in \mathbf{R}^{n-1} \times \mathbf{R} = \mathbf{R}^n$ with $|(t, \theta)|$, $|x| < 1/4$. Applying (1) and writing $(t, \theta) = \rho(y, \eta)$ where $|(y, \eta)| = 1$, we obtain

(3) $$h(x) - h(x - t) - \theta = \rho\Phi(x, \rho, y, \eta)$$

where

$$\Phi(x, \rho, y, \eta) = \sum_{i=1}^{n-1} 2k_i x_i y_i - \rho \sum_{i=1}^{n-1} k_i y_i^2 - \eta + R(x, \rho, y, \eta)$$

with

$$\begin{aligned} R(x, \rho, y, \eta) = \sum_{i,j,k=1}^{n-1} \alpha_{ijk}(x)\{&-y_i x_j x_k - y_k x_i x_j - y_j x_i x_k \\ &+ \rho x_i y_j y_k + \rho x_j y_i y_k + \rho x_k y_i y_j - \rho^2 y_i y_j y_k\} \\ + \sum_{i,j,k,l=1}^{n-1} \Bigg[\Bigg\{ y_l \int_0^1 & D_l \alpha_{ijk}(\tau x + (1 - \tau)(x - \rho y))\, d\tau \Bigg\} \\ &\times (x_i - \rho y_i)(x_j - \rho y_j)(x_k - \rho y_k) \Bigg]. \end{aligned}$$

Obviously Φ is C^∞ with domain

$$\Omega = \left\{(x, \rho, y, \eta) \colon |x| < 1/4, |\rho| < 1/4, |y|^2 + |\eta|^2 = 1\right\}.$$

For each $i = 1, 2, \ldots, (n - 1)$ we write

$$D_i\Phi = \partial\Phi/\partial x_i = 2k_i y_i + T_{i,x} + T_{i,\rho}$$

where $T_{i,x}$ stands for the sum of all terms containing at least one x_j but not ρ as a factor and $T_{i,\rho}$ stands for the sum of all terms with ρ as a factor. Since $|\alpha_{ijk}|$, $|D_l\alpha_{ijk}|$ are all bounded, by assuming $|x|$, $|\rho| < \gamma' < 1/4$ we can make $|T_{i,x}|, |T_{i,\rho}| < K_0/[2(2n)^{1/2}]$.

Also, for each $i \in \{1, 2, \ldots, n - 1\}$, by setting $x_j = 0$ for $j \neq i$ in $R(x, \rho, y, \eta)$ we will get functions of x_i, ρ, y, η in this form:

$$g_i(x_i, \rho, y, \eta)x_i^2 + \rho\phi_i(x_i, \rho, y, \eta).$$

We may assume

$$|g_i(x_i, \rho, y, \eta)|, |D_1 g_i(x_i, \rho, y, \eta)|, |D_1\phi_i(x_i, \rho, y, \eta)| \leqslant C$$

where $C > 3K(2n)^{1/2}/K_0 > 2$. Let

$$\gamma = \min\left(\gamma', 1/16, K_0/\left[4C(2n)^{1/2}\right], \rho_1\right).$$

Fix $0 < s < \gamma$ and $0 \neq w \in K_{0,s}(B) \cap \mathbf{U}_n(0, \gamma)$. Then

$$K_{0,s}(B) \cap \mathbf{U}_n(0, \gamma) = K_{0,s}(B \cap \mathbf{U}_n(0, 2\gamma)) \cap \mathbf{U}_n(0, \gamma), \tag{4}$$

$$(w - B) \cap K_{0,s} = (w + K_{0,4\gamma}) \cap K_{0,s}. \tag{5}$$

Therefore, writing $w = (t, \theta) = \rho(y, \eta)$ with $|(y, \eta)| = 1$ and using (3) we obtain

$$(w - B) \cap K_{0,s} = \left\{(x, h(x)) \cap K_{0,s}: \Phi(x, \rho, y, \eta) = 0\right\}$$

giving us (i) of Part 1.

Furthermore, if $0 \neq w \in K_{0,s}(B) \cap \mathbf{U}_n(0, \gamma)$ where $0 < s < \gamma$ and $|x| < \gamma$, then from (4) and (2) we have $0 \neq w \in \mathbf{X}(0, \infty, \mathbf{T}_0(B), 1/2)$. Since $\mathbf{T}_0(B) = \mathbf{R}^{n-1} \times \{0\}$ and $w = \rho(y, \eta)$, $|y|^2 + |\eta|^2 = 1$, there exists $i \in \{1, 2, \ldots, n - 1\}$ for which $|y_i| \geqslant [2(n-1)]^{-1/2} > (2n)^{-1/2}$. But then

$$|D_i\Phi(x, \rho, y, \eta)| = |2k_i y_i + T_{i,x} + T_{i,\rho}| \geqslant K_0/(2n)^{1/2} \geqslant 1/C.$$

Also, for $j = 1, 2, \ldots, n - 1, j \neq i$, we have

$$|D_j\Phi(x, \rho, y, \eta)/D_i\Phi(x, \rho, y, \eta)| \leqslant 3(2n)^{1/2}K/K_0.$$

This proves (ii) of Part 1.

PART 2. *Let* $0 < s < \gamma/4nC$ *and* $\rho, \eta \in \mathbf{R}$, $y \in \mathbf{R}^{n-1}$ *be given with* $|y|^2 + |\eta|^2 = 1$, $|\rho| < \gamma$ *and* $|y_i| > (2n)^{-1/2}$. *Let* $|a_i| < Csn$ *be such that* $\Phi_i(a_i, \rho, y, \eta) = 0$. *If* $|x_i| < Csn$ *and* $\Phi_i(x_i, \rho, y, \eta) = 0$, *then* $x_i = a_i$.

PROOF. Suppose $|c_i| < Csn$. Then

$$|\Phi_i(c_i, \rho, y, \eta)| = \Big|2k_i(a_i - c_i)y_i + \left[g_i(a_i, \rho, y, \eta)a_i^2 - g_i(c_i, \rho, y, \eta)c_i^2\right] + \rho\left[\phi_i(a_i, \rho, y, \eta) - \phi_i(c_i, \rho, y, \eta)\right]\Big|.$$

Applying the mean value theorem to g_i and ϕ_i and using the bounds on g_i, $D_1 g_i$, $D_1\phi_i$ one can show that since $Csn < \gamma$ this expression is not less than $|a_i - c_i|K_0/(2n)^{1/2}$.

PART 3. *If* $0 < s < \gamma/4nC$ *and*

$$0 \neq w = \rho(y, \eta) \in K_{0,s/C}(B \cap \mathbf{U}_n(0, 2\gamma)) \cap \mathbf{U}_n(0, \gamma) \quad \textit{with } |(y, \eta)| = 1,$$

then there exists $i \in \{1, 2, \ldots, n - 1\}$ (*depending only on* y) *such that* $|y_i| > (2n)^{-1/2}$. *Corresponding to such an* i *there is a* C^∞ *function*

$$\psi_{w,i}: \mathbf{U}_{n-2}(0, s/C) \to \mathbf{R}$$

such that

(i) *If* $|z| < s/C$, *then*

$$\Phi(\sigma_i(z), \rho, y, \eta) = 0$$

where we define

$$\sigma_i(z) = (z_1, \dots, z_{i-1}, \psi_{w,i}(z), z_i, \dots, z_{n-2}) \quad (\psi_{w,i}(z) \textit{ in ith place}).$$

(ii)

$$|\psi_{w,i}(0)| < 2ns.$$

(iii) *If* $x \in \mathbf{R}^{n-1}$, $|x| < s/C$ *and* $\Phi(x, \rho, y, \eta) = 0$, *then*

$$x = \sigma_i(\pi_i x).$$

It follows that

$$\mathbf{H}^{n-2}[(w - B) \cap K_{0,6ns}] \geqslant H_1 s^{n-2}$$

where $H_1 = C^{2-n}\mathbf{H}^{n-2}[\mathbf{U}_{n-2}(0, 1)]$.

PROOF. We have $w = g_0 + b_0$ where $g_0 \in K_{0,s/C}$ and $b_0 \in B \cap \mathbf{U}_n(0, 2\gamma)$. Set $g_0 = (x_0, h(x_0))$ where $x_0 \in \mathbf{R}^{n-1}$. By Part 1(i), $\Phi(x_0, \rho, y, \eta) = 0$. In view of (ii) of Part 1, we may assume that $|y_{n-1}| > (2n)^{-1/2}$ and $|D_{n-1}\Phi(x, \rho, y, \eta)| \geqslant 1/C$ for $|x| < \gamma$. We will identify $\mathbf{R}^{n-2} \times \mathbf{R} = \mathbf{R}^{n-1}$; denote $x_0 = (z_0, \zeta_0)$. By the implicit function theorem there exist a $\delta > 0$ and a C^∞ function ψ_δ: $\mathbf{U}_{n-2}(z_0, \delta) \to \mathbf{R}$ such that $\psi_\delta(z_0) = \zeta_0$ and $\Phi((z, \psi_\delta(z)), \rho, y, \eta) = 0$, $|(z, \psi_\delta(z))| < \gamma/2$ for $|z - z_0| < \delta$, and the relations

$$(*) \quad \Phi(x', \rho, y, \eta) = 0, \quad |x'_{n-1} - \psi_\delta(\pi_{n-1}x')| < \delta, \quad |\pi_{n-1}x' - z_0| < \delta$$

hold only in case $x'_{n-1} = \psi_\delta(\pi_{n-1}x')$. Since ρ, y, η are all fixed we may write $\Phi(z, \psi_\delta(z))$ in place of $\Phi((z, \psi_\delta(z)), \rho, y, \eta)$. We claim that δ may be assumed to be greater than $2s/C$.

Let S be the set of all $\delta > 0$ corresponding to which there exists ψ_δ as above and let $\delta_0 = \text{lub } S$. We may assume $\delta_0 \leqslant 2s/C$. Since the ψ_δ are unique we conclude that $\delta_0 \in S$ with $\cup\{\psi_\delta: \delta \in S\} = \psi_{\delta_0}$. Set $\psi = \psi_{\delta_0}$. By Part 1(ii), $|D_j\psi| \leqslant C, j \in \{1, 2, \dots, n-2\}$. Thus ψ is Lipschitz, and hence uniformly continuous in $\mathbf{U}_{n-2}(z_0, \delta_0)$. Therefore ψ has a continuous extension to the Closure of $\mathbf{U}_{n-2}(z_0, \delta_0)$. Let z be such that $|z - z_0| = \delta_0$ and denote $\zeta = \psi(z)$. Since Φ is continuous, $\Phi(z, \zeta) = 0$. By the mean value theorem and continuity of ψ, $|\zeta| < 2ns$. Also $|z| < 3s$, so $|(z, \zeta)| \leqslant 4ns < \gamma$. Thus $D_{n-1}\Phi(z, \zeta) \neq 0$. Hence ψ has a C^∞ extension in a neighborhood of the compact set Clos $\mathbf{U}_{n-2}(z_0, \delta_0)$ [H, p. 23] which contradicts the maximality of δ_0. Writing $\psi_{w,i} = \psi$ we conclude that (i) and (ii) hold. Note that it also follows from what we have shown that

$$((z, \zeta), h(z, \zeta)) \in K_{0,6ns} \quad \text{for } |z| \leqslant s/C \text{ and } \psi(z) = \zeta.$$

Thus by Part 1(i) we conclude that

$$\mathbf{H}^{n-2}\big[(w - B) \cap K_{0,6ns}\big]$$
$$\geqslant \mathbf{H}^{n-2}\big[P_{n-1}((w - B) \cap K_{0,6ns})\big]$$
$$\text{where } P_{n-1}(x_1, \ldots, x_{n-1}, x_n) = (x_1, \ldots, x_{n-1}) \in \mathbf{R}^{n-1}$$
$$\geqslant \mathbf{H}^{n-2}\big[\mathbf{U}_{n-2}(0, 1)\big](s/C)^{n-2}.$$

Finally, fix $x \in \mathbf{R}^{n-1}$ such that $|x| < s/C$ and $\Phi(x, \rho, y, \eta) = 0$. Choosing $\psi'_{w,n-1}$ as above with x_0 replaces by x we infer from Part 2 that $\psi_{w,n-1}(0) = \psi'_{w,n-1}(0)$. Thus by $(*)$

$$\psi'_{w,n-1}|\mathbf{U}_{n-2}(0, s/C) = \psi_{w,n-1}|\mathbf{U}_{n-2}(0, s/C)$$

which proves (iii).

PART 4. *There exist* $0 < \alpha_0 < 1$, $0 < s_0 < \gamma/4nC$, $0 < \delta_0 < \gamma$ *and for each* $i \in \{1, 2, \ldots, n-1\}$ *a positive integer* m_i *such that the following is true*:

Let

$$W = \mathbf{R}^{n-1} \times \mathbf{R} \cap \big\{(y, \eta): |(y, \eta)| = 1, |\eta| \leqslant \alpha_0 s_0\big\}.$$

For each $i \in \{1, 2, \ldots, n-1\}$ *and* $j \in \{1, 2, \ldots, m_i\}$ *there exist* $\delta_{ij} > 0$, $(y_{ij}, \eta_{ij}) \in W_i = W \cap \{(y, \eta)\colon |(y, \eta)| = 1, |y_i| \geqslant [2(n-1)]^{-1/2}\}$ *and* θ_{ij}: $\mathbf{R}^{n-2} \cap \{z\colon |z| < \delta_{ij}\} \times \mathbf{R} \cap \{\rho\colon |\rho| < \delta_{ij}\} \times (\mathbf{R}^{n-1} \times \mathbf{R}) \cap \{(y, \eta)\colon |(y, \eta)| = 1, |(y, \eta) - (y_{ij}, \eta_{ij})| < \delta_{ij}, |y_i| > (2n)^{-1/2}\} \to \mathbf{R}$ *all having the same Lipschitz constant* H_0, *such that for each* $(y_0, \eta_0) \in W$ *there exist i and j for which*:

(i) $\mathbf{R}^{n-1} \times \mathbf{R} \cap \{(y, \eta)\colon |(y, \eta)| = 1, |(y, \eta) - (y_0, \eta_0)| < \delta_0\} \subset \mathbf{R}^{n-1} \times \mathbf{R} \cap \{(y, \eta)\colon |(y, \eta)| = 1, |(y, \eta) - (y_{ij}, \eta_{ij})| < \delta_{ij}, |y_i| > (2n)^{-1/2}\}$.

(ii) *If* $|\rho| < \delta_0$, $(y, \eta) \in \mathbf{R}^{n-1} \times \mathbf{R} \cap \{(y', \eta')\colon |(y', \eta')| = 1, |(y', \eta') - (y_0, \eta_0)| < \delta_0\}$ *and* $|z| \in \mathbf{R}^{n-2}$ *with* $|z| < \delta_0$, *then*

$$\big|\theta_{ij}(z, \rho, y, \eta)\big| < C s_0 n,$$

and

$$\Phi\big(\sigma_{ij}(z, \rho, y, \eta), \rho, y, \eta\big) = 0$$

where we define

$$\sigma_{ij}(z, \rho, y, \eta) = \big(z_1, \ldots, \theta_{ij}(z, \rho, y, \eta), \ldots, z_{n-2}\big) \quad (\theta_{ij} \textit{ in ith place}).$$

PROOF. Using differentiability of h together with Lemma 1 and the fact that $0 \in B$, we infer the existence of $0 < s_0 < \gamma/4nC$, and $0 < \alpha_0 < 1$ such that

$$\operatorname{Clos} \mathbf{X}(0, \gamma/2, \mathbf{T}_0(B), \alpha_0 s_0) \subset K_{0,s/C}(B) \cap \mathbf{U}_n(0, \gamma)$$
$$\subset \mathbf{X}(0, \infty, \mathbf{T}_0(B), 1/2) \cup \{0\}.$$

We may assume that $i = n - 1$; fix $(y, \eta) \in W_{n-1}$ and let $0 < |\rho| < \gamma/2$. Then

$$0 \neq \rho(y, \eta) \in \operatorname{Clos} \mathbf{X}(0, \gamma/2, \mathbf{T}_0(B), \alpha_0 s_0) \subset K_{0,s/C}(B) \cap \mathbf{U}_n(0, \gamma).$$

Let $\psi_{\rho(y,\eta),n-1}$ be the C^∞ function found in Part 3. Observing that i is independent of ρ, we conclude using Part 3(i), (ii) and continuity of Φ at $\rho = 0$ that there exists $a_{n-1} \in \mathbf{R}$ such that $\Phi((0, a_{n-1}), 0, y, \eta) = 0$ and $|a_{n-1}| \leq 2ns < Cns_0$. Also we infer from Part 1(ii) that $|D_{n-1}\Phi((0, a_{n-1}), 0, y, \eta)| \geq 1/C$. Using the implicit function theorem at each of the points $((0, a_{n-1}), 0, y, \eta) \in \mathbf{R}^{n-1} \times \mathbf{R} \times \mathbf{R}^{n-1} \times \mathbf{R}$ where $(y, \eta) \in W_{n-1}$ together with the Lebesgue covering lemma, we easily infer (i) and (ii) by noting that if $(y_0, \eta_0) \in W$ then

$$(y_0, \eta_0) \in \operatorname{Clos} \mathbf{X}(0, 1, \mathbf{T}_0(B), \alpha_0 s_0) \subset \mathbf{X}(0, \infty, \mathbf{T}_0(B), 1/2) \cup \{0\};$$

consequently there exists an $i \in \{1, 2, \ldots, n-1\}$ such that

$$|y_{0i}| \geq [2(n-1)]^{-1/2}$$

and so $(y_0, \eta_0) \in W_i$.

PART 5. *There exists* $0 < \beta_0 < 1$ *such that if* $0 < s < \min(\beta_0 s_0, \delta_0)$, $0 < r < \delta_0$ *and* $0 \neq w_0 = \rho_0(y_0, \eta_0) \in K_{0,s/C}(B) \cap \mathbf{U}_n(0, r)$ *with* $|(y_0, \eta_0)| = 1$, *then there exist* $i \in \{1, 2, \ldots, n-1\}$ *and* $j \in \{1, 2, \ldots, m_i\}$ *such that whenever* $0 \neq w = \rho(y, \eta) \in K_{0,s/C}(B) \cap \mathbf{U}_n(0, r)$ *with* $|(y, \eta)| = 1$ *and* $|(y, \eta) - (y_0, \eta_0)| < \delta_0$, *we have*

$$K_{0,s/C} \cap (w - B) \subset \{(\sigma_{ij}(z, \rho, y, \eta), h \circ \sigma_{ij}(z, \rho, y, \eta)) : |z| \leq s/C\}.$$

It follows that

$$\mathbf{H}^{n-2}[K_{0,s/C} \cap (w - B)] \leq H_2 (s/C)^{n-2}$$

where H_2 *is a real number which does not depend on* s, w *and* w_0.

PROOF. We note that there exists $0 < \beta_0 < 1$ such that

$$K_{0,\beta_0 s_0/C}(B) \cap \mathbf{U}_n(0, \gamma/2) \subset \operatorname{Clos} \mathbf{X}(0, \gamma/2, \mathbf{T}_0(B), \alpha_0 s_0).$$

Therefore $(y_0, \eta_0) \in W$; choosing $i \in \{1, 2, \ldots, n-1\}$ and $j \in \{1, 2, \ldots, m_i\}$ by Part 4(i) we see that if $0 \neq w = \rho(y, \eta) \in K_{0,s/C}(B) \cap \mathbf{U}_n(0, r)$ with $|(y, \eta)| = 1$ and $|(y, \eta) - (y_0, \eta_0)| < \delta_0$, then $|y_i| > (2n)^{-1/2}$. Thus if $(x, h(x)) \in K_{0,s/C} \cap (w - B)$, then by Parts 1 and 3 there exists a C^∞ function $\psi_{w,i}$ such that

$$x = \sigma_i(\pi_i x) = (x_1, \ldots, \psi_{w,i}(\pi_i x), \ldots, x_{n-1}).$$

We infer from Part 4(ii) and Part 2 that we must have

$$\theta_{ij}(\pi_i x, \rho, y, \eta) = \psi_{w,i}(\pi_i x),$$

hence $x = \sigma_{ij}(\pi_i x, \rho, y, \eta)$. The desired inclusion follows.

Now the map F defined by

$$F(x) = (x, h(x)), \qquad x \in U_0,$$

is Lipschitzian since h is Lipschitzian on U_0; let M be a Lipschitz constant for F. Furthermore, $1 + H_0$ is a Lipschitz constant for σ_{ij}. Therefore,

$$\mathbf{H}^{n-2}\left[K_{0,s/C} \cap (w - B)\right]$$

$$< M^{n-2}\mathbf{H}^{n-2}\left[\mathbf{R}^{n-1} \cap \left\{\sigma_{ij}(z, \rho, y, \eta) \colon |z| < s/C\right\}\right] < H_2(s/C)^{n-2}$$

where $H_2 = [M(1 + H_0)]^{n-2}\mathbf{H}^{n-2}[\mathbf{U}_{n-2}(0, 1)]$. This completes the proof of Part 5.

Combining Part 3 and Part 5 and noting (4) we get the desired lemma.

LEMMA 3. *Let A be a Suslin subset of $\mathbf{R}^n$. Then $\mathbf{H}^{n-1}$ almost all $g \in -B$ satisfy one of the following three conditions*:

(i) *For some $\varepsilon > 0$,*

$$\limsup_{s \to 0^+} \sup_{0 < r < \varepsilon} (rs)^{-1}\phi\left[A \cap K_{g,s}(B) \cap \mathbf{U}_n(0, r) \sim \{0\}\right] = 0.$$

(ii) *For all $\varepsilon > 0$,*

$$\limsup_{s \to 0^+} \sup_{0 < r < \varepsilon} (rs)^{-1}\phi\left[A \cap K_{g,s}(B) \cap \mathbf{U}_n(0, r) \sim \{0\}\right] = \infty.$$

(iii) $0 \in \mathrm{Clos}[A \cap (g + B) \sim \{0\}]$.

PROOF. The proof is similar to that of [**B**, 3.7].

Let r_2 be as in Lemma 2. We may assume $A \subset \mathbf{U}_n(0, r_2)$. Consider the map $F\colon -B \times B \to \mathbf{R}^n \times -B$ with $F(g, b) = (g + b, g)$. Denote $\Phi = F(-B \times B)$. Also, let

$$\pi_1\colon \mathbf{R}^n \times (-B) \to \mathbf{R}^n \quad \text{with } \pi_1(u, g) = u$$

and

$$\pi_2\colon \mathbf{R}^n \times (-B) \to -B \quad \text{with } \pi_2(u, g) = g$$

denote the projection maps.

For $u \in \mathbf{R}^n$ denote $\Phi_u = \pi_1^{-1}\{u\} \cap \Phi$. Thus

$$\Phi_u = \{(u, g) \colon g \in (u - B) \cap (-B)\},$$

and it follows that for $g \in -B$ and $s > 0$

$$\Phi_u \cap \pi_2^{-1}(K_{g,s}) = \{(u, h) \colon h \in K_{g,s} \cap (u - B)\}.$$

Let ϕ' be the measure on Φ such that for $S \subset \Phi$,

$$\phi'(S) = \int_A^* \mathbf{H}^{n-2}(\Phi_u \cap S)\, d\phi u.$$

Now, for each positive integer ν we introduce the measure Ψ_ν over $(-B)$ defined by

$$\Psi_\nu(T) = \sup_{0 < r < 1/\nu} \phi'\left[\pi_1^{-1}(\mathbf{U}_n(0, r)) \cap \pi_2^{-1}(T) \cap \Phi\right] r^{-1}$$

for $T \subset -B$. Let

$$P_\nu = (-B) \cap \left\{g\colon \lim_{s\to 0^+}\left[\Psi_\nu(K_{g,s})/s^{n-1}\right] = 0\right\},$$

$$Q_\nu = (-B) \cap \left\{g\colon \lim_{s\to 0^+}\left[\Psi_\nu(K_{g,s})/s^{n-1}\right] = \infty\right\},$$

$$R_\nu = \pi_2\left[\pi_1^{-1}\left(A \cap \mathbf{U}_n(0, \nu^{-1})\right) \cap \Phi\right],$$

$$P = \bigcup_{\nu=1}^{\infty} P_\nu, \quad Q = \bigcap_{\nu=1}^{\infty} Q_\nu, \quad R = \bigcap_{\nu=1}^{\infty} R_\nu.$$

It follows that $\mathbf{H}^{n-1}[(-B) \sim (P \cup Q \cup R)] = 0$.

Using Lemma 2 we complete the proof by proceeding as in the proof of [**B**, 3.7].

LEMMA 4. *Let A be a Suslin subset of $\mathbf{R}^n$ with $\phi(A) < \infty$. Then*

$$\phi \times \mathbf{H}^{n-1}\left[A \times B \sim (S_{A,1} \cup S_{A,2} \cup S_{A,3})\right] = 0.$$

PROOF. Let us fix $a \in \mathbf{R}^n$ and $(a, b) \in \{a\} \times B$. We note that $\tau_{-a}(A)$ is a Suslin set. Also $(\tau_{-a})_\# \phi$ is a nonnegative measure such that closed sets are $(\tau_{-a})_\# \phi$ measurable. Consequently, replacing A and ϕ in Lemma 3 by $\tau_{-a}(A)$ and $(\tau_{-a})_\# \phi$ we see that

$$\mathbf{H}^{n-1}\left[\{a\} \times B \sim S_{A,1} \cup S_{A,2} \cup S_{A,3}\right] = 0.$$

We infer from [**B**, 4.1–4.2] that $S_{A,1}$, $S_{A,2}$, $S_{A,3}$ are Suslin sets. Also, $\phi(A) < \infty$, hence we can apply Fubini's theorem to obtain our assertion.

LEMMA 5. *If A is a purely $(\phi, 1)$ unrectifiable Suslin subset of $\mathbf{R}^n$ such that $\phi(A) < \infty$ and $\phi(W) = 0$ whenever $W \subset A$ and $\mathbf{H}^1(W) = 0$, then*

$$\phi \times \mathbf{H}^{n-1}\left[A \times B \cap S_{A,1}\right] = 0.$$

PROOF. This is the special case of [**B**, 4.6] where $G = X = \mathbf{R}^n$.

LEMMA 6. *If A is a Suslin subset of $\mathbf{R}^n$ and $\phi(A) < \infty$, then*

$$\mathbf{H}^n\left[f_0(A \times B \cap S_{A,2})\right] = 0.$$

PROOF. This is immediate from [**B**, 4.7] with $G = X = \mathbf{R}^n$.

LEMMA 7. *Let A be a Suslin subset of $\mathbf{R}^n$ with $\mathbf{H}^1(A) < \infty$. Then*

$$\mathbf{H}^n\left[f_0(A \times B \cap S_{A,3})\right] = 0.$$

PROOF. Since B is separable it is sufficient to show that

$$\mathbf{H}^n\left[f_0(A \times B_0 \cap S_{A,3})\right] = 0$$

where $B_0 = B \cap U_0$, U_0 being an open subset of $\mathbf{R}^n$ with $\mathbf{H}^{n-1}(B_0) < \infty$.

Now, by [F1, 2.10.45] and [F1, 2.10.25] be conclude that

$$\int^{*}_{\mathbf{R}^n} \mathbf{H}^0\big(A \times B_0 \cap f_0^{-1}\{g\}\big)\, d\mathbf{H}^n g \leqslant C_1 \mathbf{H}^n(A \times B_0) < \infty$$

where C_1 is a positive constant. Thus

$$\mathbf{H}^n\Big[\mathbf{R}^n \cap \big\{g\colon \mathbf{H}^0\big(A \times B_0 \cap f_0^{-1}\{g\}\big) = \infty\big\}\Big] = 0,$$

and it is not difficult to show that

$$f_0[A \times B_0 \cap S_{A,3}] \subset \big\{g\colon \mathbf{H}^0\big(A \times B_0 \cap f_0^{-1}\{g\}\big) = \infty\big\}.$$

THEOREM 1. *Suppose* $E \subset \mathbf{R}^n$ *with* $\mathbf{H}^1(E) < \infty$. *Then there exists a countably* 1-*rectifiable Borel subset* R *of* $\mathbf{R}^n$ *such that* $(E \sim R)$ *is purely* $(\mathbf{H}^1, 1)$ *unrectifiable and*

$$(E \sim R) \cap (g + B) = \varnothing$$

for $\mathbf{H}^n$ *almost all* $g \in \mathbf{R}^n$.

Furthermore, if in addition E *is* $\mathbf{H}^1$ *measurable and* $E \cap (g + B) = \varnothing$ *for* $\mathbf{H}^n$ *almost all* $g \in \mathbf{R}^n$, *then* $\mathbf{H}^1(E \cap R) = 0$, *hence* E *is purely* $(\mathbf{H}^1, 1)$ *unrectifiable.*

PROOF. Since $\mathbf{H}^1$ is Borel regular, we may assume E to be Borel. By maximizing the finite measure $\mathbf{H}^1 \llcorner E$ on the class of countably 1-rectifiable Borel subsets of $\mathbf{R}^n$ and using [F1, 3.2.14] we obtain a countably 1-rectifiable Borel subset R of $\mathbf{R}^n$ such that $A = (E \sim R)$ is purely $(\mathbf{H}^1, 1)$ unrectifiable. Applying [B, 4.1 and 4.2] with $G = X = \mathbf{R}^n$ we infer that $S_{A,1}$, $S_{A,2}$, $S_{A,3}$ are Suslin sets. Using Lemmas 4–7 together with [F1, 2.10.25] we easily conclude that

$$\mathbf{H}^n\big[f_0(A \times B)\big] = 0$$

which is equivalent to $A \cap (g + B) = \varnothing$ for $\mathbf{H}^n$ almost all $g \in \mathbf{R}^n$.

Now let E be $\mathbf{H}^1$ measurable with $E \cap (g + B) = \varnothing$ for $\mathbf{H}^n$ almost all $g \in \mathbf{R}^n$. Observe that by [F1, 3.2.28] we may assume R to be a subset of a proper 1-dimensional submanifold R_0 of class 1 of $\mathbf{R}^n$. We may also assume $0 \in R$ and $\mathbf{R}e_n = \mathbf{T}_0(R_0)$ where $\{e_1, \dots, e_n\}$ is the standard orthonormal basis for $\mathbf{T}_0(\mathbf{R}^n) = \mathbf{R}^n$. Let

$$M = \mathbf{S}^{n-1} \cap \{u\colon u \cdot e_n = 0\}.$$

Assuming B is oriented with a unit normal vector field ν, let us consider the Gauss map $\eta\colon B \to \mathbf{S}^{n-1}$ defined by $\eta(b) = \nu(b) \in \mathbf{R}^n$ for $b \in B$. Since the Gaussian curvature is nonzero at every point of B, η has nonzero Jacobian at every point. Therefore by the inverse function theorem $\eta(B)$ is an open subset of $\mathbf{S}^{n-1}$ and thus $\mathbf{H}^{n-1}[\eta(B)] > 0$. There therefore exists $b \in B$ at which $\eta(b) \not\in M$. We can assume $b = 0$; thus $e_n \not\in T_0(B)$ and we see that $\mathbf{R}e_n + \mathbf{T}_0(B) = \mathbf{R}^n$.

For $r > 0$ denote $B_r = B \cap \mathbf{U}_n(0, r)$. Since $(E \cap R) \times B_r$ is $\mathbf{H}^n$ measurable application of [**F1**, 3.2.3] gives

$$\int_{(E\cap R)\times B_r} J(f_0|R_0 \times B)\, d\mathbf{H}^n = \int_{\mathbf{R}^n} \mathbf{H}^0\Big[(f_0|(E \cap R_0) \times B_r)^{-1}\{g\}\Big]\, d\mathbf{H}^n g.$$

Now the integral on the right is zero by our hypothesis. Moreover, if $\{u_1, \ldots, u_{n-1}\}$ is an orthonormal basis of $\mathbf{T}_0(B)$ then

$$J(f_0|R_0 \times B)(0, 0) = |e_n \wedge u_1 \wedge \cdots \wedge u_{n-1}| > 0.$$

Consequently, $\mathbf{H}^n[(E \cap R) \times B_r] = 0$ for some $r > 0$, whence we conclude using [**F1**, 3.2.25] that $\mathbf{H}^1(E \cap R) = 0$.

If $u \in \mathbf{R}^n$, the k-dimensional upper density of ϕ at u is

$$\theta^{*k}(\phi, u) = \limsup_{r\to 0^+} \alpha(k)^{-1} r^{-k} \phi(\mathbf{U}_n(u, r))$$

where $\alpha(k)$ is the volume of the unit k-ball $\mathbf{U}_k(0, 1)$.

THEOREM 2. *Suppose $W \subset \mathbf{R}^n$, $\phi(W) < \infty$, $\phi(S) = 0$ whenever $S \subset W$ and $\mathbf{H}^1(S) = 0$ and $\theta^{*1}(\phi \llcorner W, u) > 0$ for ϕ almost all $u \in W$. Then there exists a countably $(\phi, 1)$-rectifiable and ϕ measurable set Q such that $(W \sim Q)$ is purely $(\phi, 1)$ unrectifiable and*

$$(W \sim Q) \cap (g + B) = \varnothing$$

for $\mathbf{H}^n$ almost all $g \in \mathbf{R}^n$.

Furthermore, if in addition W is a Borel set such that $W \cap (g + B) = \varnothing$ for $\mathbf{H}^n$ almost all $g \in \mathbf{R}^n$, then W is purely $(\phi, 1)$ unrectifiable.

PROOF. Applying Theorem 1 one proceeds in a manner similar to the proof of [**B**, 5.3].

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